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Convergence of Lattice Gauge Theory for Maxwell's equation

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Abstract

In this article we show convergence of Lattice Gauge Theory with gauge group U(1) in the energy norm. This is done by stability analysis and comparison with the classical Yee-scheme which is convergent.

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1 Introduction

Almost every physical theory has a variational formulation, and Maxwell's equations, describing electromagnetism, are no exception. The Langrangian which describes Maxwell's equations is a special type of the class of Yang-Mills Langrangians. The Yang-Mills Langrangians are functions which are not only relativistic, i.e. Lorentz invariant, but also invariant under gauge transformations, which are a special type of internal symmetries.

Both of these symmetries are difficult to preserve under a discretization, independent of whether one uses a Finite Element approach or a Finite Difference approach. For that reason, interest is present on how to transfer these symmetries to the discrete level.

The problem concerning the gauge symmetry, has in the simplified case with flat space-time been solved by Lattice Gauge Theory [1, 2, 3]. This is a method from physics, which was created to remove unfavourable divergences in high energy physics, often called ultraviolet divergences, and at the same time respect the gauge invariance (the Lorentz symmetry is not preserved).

The LGT was initially developed to calculate quantities in the SU(3) part of the Standard Model in physics (the model describing the fundamental particles and the fundamental forces of nature, except gravity), but the model is equally well suited for describing the Maxwell part, the U(1) part.

However, LGT may at first glance appear as a brutal approximation to pure electromagnetism. Although LGT respects the local gauge invariance, it produces a set of nonlinear difference equations approximating the linear Maxwell's equations. In addition, the popular Yee scheme in a second order formulation, an explicit Finite Difference scheme, is both linear and locally gauge invariant.

In spite of this, LGT introduces some major advantages. Probably the greatest achievement of LGT is how it approximates the covariant derivative in nonlinear wave equations. In standard finite difference approximations, non-local terms arise which cannot be locally gauge invariant. This nonlocality is resolved by LGT by connecting the nonlocal terms together by a connection. This prescription can be used in the construction of the covariant derivative in the continuous case [4], and is of course the inspiration of LGT. Examples where LGT can be sucessfull are the Maxwell-Klein-Gordon equation, the Maxwell-Dirac equation, the Yang-Mills-Higgs equation etc.

In a previous article [5] we have studied the Maxwell-Klein-Gordon equation with a numerical scheme consisting of a Yee-scheme [6] for the Maxwell part and an LGT scheme for the Klein-Gordon part. This numerical scheme has some nice properties, and is for instance locally gauge invariant, implying that the scheme is charge conserving. As a first step towards convergence analysis of this scheme and ultimately of LGT for the Yang-Mills-Higgs equation we will in this article study the simplest possible version of LGT, i.e. LGT for Maxwell theory. We will give a short introduction to the model and then prove convergence by comparison with the classical Yee-scheme.

The paper is organized as follows: In §2 we introduce continuous Maxwell theory both in a first and second order formulation. §3 is devoted to LGT and its finite difference equations. In §4 the Yee-scheme is briefly discussed. In §5 convergence of the LGT-scheme is shown by a comparison with the Yee-scheme. In §6 we present some numerical results. Finally in §7 some concluding remarks are drawn.

2 Continuous Maxwell theory

Maxwell's equations[1] (in Heaviside-Lorentz units) are

$$\begin{aligned} \mathrm{div}\mathbf{E} &= \rho \\ \mathrm{curl}\mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} &= \mathbf{j} \\ \mathrm{div}\mathbf{B} &= 0 \\ \mathrm{curl}\mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0 \end{aligned} \tag{1}$$

with $\mathbf{E}$ and $\mathbf{B}$ as the electric and magnetic field strengths and ρ and $\mathbf{j}$ as the charge density and current density. In a covariant formulation the electric and magnetic fields are combined in the Electromagnetic field tensor F ($= F_{\mu\nu} dx^\mu \wedge dx^\nu$ in a coordinate basis, where the indices μ and ν run from 0 to 3. $\mu = 0$ represents the time component of the space-time coordinates and $\mu = 1, 2, 3$ the space components.), which is the space-time exterior derivative, $d = d_t + d_{\mathbf{x}}$, of a gauge potential $x = (t, \mathbf{x}) \mapsto A_0(x)\mathrm{d}t + \mathbf{A}(x) = A_\mu(x)$, where A_0 is a real-valued function and $\mathbf{A}$ is a real-valued one-form, i.e. $F = dA$. The electric and magnetic fields are identified as (we are using the Minkowski space-time metric $\eta_{\mu\nu}$ =diag(-1,1,1,1) to raise and lower indices, but it holds with a general metric as well)

$$\mathbf{E} = -\dot{\mathbf{A}} - \mathrm{grad} A^0, \qquad \mathbf{B} = \mathrm{curl}\mathbf{A}. \tag{2}$$

The dual field tensor is defined by $\tilde{F} = \star F$, where $\star$ is the Hodge star operator, which is a linear transformation from the space of 2-forms to the space of (4-2)-forms defined by the metric. When written in terms of the field tensors and the 4-vector current density $J = (\rho, \mathbf{j})$ Maxwell's equations get the following compact form

$$\begin{aligned} dF &= 0 \qquad \text{Bianchi identity} \\ d\tilde{F} &= J. \end{aligned} \tag{3}$$

We immediately see that the four-current has to be divergence free, i.e. $dJ = 0$, due to the identity $\mathrm{d}^2 = 0$.

2.1 The variational formulation of Maxwell's equations

Maxwell's equations can be derived by a variational principle from the following action functional

$$S[A] = \int_\Omega dt d^3x \mathcal{L}(A, dA), \tag{4}$$

where $\mathcal{L}$ is the Lagrangian density and Ω is the chosen space-time region. The Lagrangian density is a local function of the field variable A and its exterior derivative

$$\mathcal{L} = -\frac{1}{2} dA \cdot dA + J \cdot A = \frac{1}{2}(\mathbf{E} \cdot \mathbf{E} - \mathbf{B} \cdot \mathbf{B}) + J \cdot A, \tag{5}$$

with $\cdot$ representing the space-time scalar product determined by the metric.

A solution of Maxwell's equations for given boundary conditions corresponds to a solution of the variational equation

$$\delta S = 0, \tag{6}$$

[1] This section follows closely the lecture notes *Non-Relativistic Quantum Mechanics* by Jon Magne Leinaas [7]

where this condition should be satisfied for arbitrary variations in the field variables, with fixed values on the boundary of Ω. By solving Eq. 6 corresponds to solving the Euler-Lagrange equation in a coordinate basis (in this article we are using Einsten summation convention, meaning that a summation over repeated indices is assumed)

$$\frac{\partial \mathcal{L}}{\partial A^\mu} - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial^\nu A^\mu)} \right) = 0, \qquad \forall \mu. \tag{7}$$

With $\mathcal{L}$ given by (5) it is straightforward to check that the inhomogeneous Maxwell equations are reproduced by the Euler-Lagrange equations.

2.2 Gauge invariance

We see that the physical fields $\mathbf{E}$ and $\mathbf{B}$ are unaltered when the electromagnetic potentials are transformed as

$$A \rightarrow A' = A + d\lambda, \tag{8}$$

where λ is a scalar function of space and time. This is called gauge invariance, and the usual way to view the invariance of the fields under this transformation is that it reflects the presence of a non-physical degree of freedom in the potentials. The potentials define an overcomplete set of variables for the electromagnetic field.

From (1) we see that Maxwell's equations can be divided into two evolution equations

$$\begin{aligned} \text{curl}\mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} &= \mathbf{j} \\ \text{curl}\mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0 \end{aligned} \tag{9}$$

and two constraint equations

$$\begin{aligned} \text{div}\mathbf{E} &= \rho \\ \text{div}\mathbf{B} &= 0. \end{aligned} \tag{10}$$

An important result regarding the constraints is

Theorem 1 *Suppose* $(\mathbf{E}, \mathbf{B})$ *solves equation* (9) *on a time interval [0,T]. Suppose furthermore that the constraints* (10) *are satisfied at* $t = 0$ *and that the four-current is divergence free, i.e.* $dJ = 0$. *Then the constraints* (10) *are satisfied for all* $t \in [0, T]$.

-Proof Differentiate the constraints (10) with respect to time, and use equation (9) to get the conservation.

■

The above result is a statement about charge conservation and can be seen as a direct consequence of the local gauge invariance. The connection can be made explicit through Noether's theorem, which states that for every continuous symmetry there is a conserved quantity [8, 9, 10].

Because of the equivalence between charge conservation and gauge invariance, the gauge invariance is regarded as a fundamental property of the theory, and can be viewed as an analogue of the equivalence principle of general relativity.

2.3 Energy

The energy density of the electromagnetic field is given by

$$\mathcal{H} = \frac{1}{2}(\mathbf{E}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{B}). \tag{11}$$

A direct calculation shows that

$$\frac{d\mathcal{H}}{dt} = -\mathbf{j}\cdot\mathbf{E} - \operatorname{div}(\mathbf{E}\times\mathbf{B}). \tag{12}$$

This means that with no current density, i.e. $\mathbf{j} = 0$, the total electromagnetic energy

$$H = \int d^3x \frac{1}{2}(\mathbf{E}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{B}) = \frac{1}{2}(\|\mathbf{E}\|_{L^2}^2 + \|\mathbf{B}\|_{L^2}^2) \tag{13}$$

is conserved (since we assume as always that the fields fall off sufficiently rapidly at infinity, alternatively that the fields are zero on the boundary of a bounded domain or periodic boundary conditions).

3 Lattice Gauge Theory for Maxwell theory

Lattice Gauge Theory (LGT) is a numerical method, originally from physics, developed for studying gauge theories on a space-time that has been discretized on a hypercubic lattice [1, 2, 3, 11, 12, 13]. In the field of physics LGT is particulary popular in the QCD-part (describing the nuclear force) of the Standard model, where perturbation calculations are troublesome. However, the model is equally well defined for other gauge theories, especially Maxwell theory.

The strength of LGT shows off most clearly when the gauge fields are coupled to either a complex scalar field or a fermionic field. What LGT provides is a structure preserving discretization of the covariant derivative which couples the gauge fields to the scalar/fermionic field. The covariant derivative applied to a scalar/fermionic field should transform in the same way as the field itself, and when using a standard finite difference approximation this is impossible due to non-local terms. LGT approximates the covariant derivative by parallel transport of the fields in neighbouring points by the gauge fields, and this makes the approximation local, hence it will transform in the right way.

The case with a complex scalar field coupled to the U(1) gauge field, i.e. Maxwell-Klein-Gordon theory, has been studied by the authors in [5].

3.1 The Wilson loop

In the continuous theory, terms that are non-local need to be modified in order to be gauge invariant [4]. The way this is done is by using the transformation property of the Wilson line defined by

$$U(x, y) = e^{i\int_P A(z)\mathrm{d}z}, \tag{14}$$

where P is a path between x and y. When the gauge group is non-commuting, the integral should be path ordered. We see that under a local U(1) gauge transformation, where the gauge field, living in the adjoint representation, transforms as $A \to A + d\lambda$, the Wilson line transforms as

$$U(x, y) \mapsto G^{-1}(x)U(x, y)G(y), \qquad G(x) = e^{i\lambda(x)}. \tag{15}$$

If $x = y$, the path P is a closed loop, $U(x,x)$ is called the Wilson loop, and we see that it transforms as $U(x,x) \mapsto G^{-1}(x)U(x,x)G(x)$, so it is gauge invariant when the gauge group is commuting. In the Maxwell case, with gauge group U(1), the Wilson loop can be rewritten by Stokes' theorem as

$$U(x,x) = e^{i\oint_P A(z)\mathrm{d}z} = e^{i\frac{1}{2}\int_\Sigma F d\sigma} \tag{16}$$

where Σ is the surface that spans the closed loop P, $d\sigma$ is an area element on this surface, and F is the field tensor. Since the Wilson loop is gauge invariant, this visualizes the gauge invariance of the field strenght.

With this in mind we are ready to construct a gauge invariant action for the kinetic Maxwell part [2]. We are considering a hypercubic lattice with lattice points $n = (n_t, \mathbf{n})$ and lattice spacing h in the spatial directions and Δt in the temporal direction. To each set of neighbouring points on the lattice, n and $n + a_\mu \mathbf{e}_\mu := n + a_\mu$ where $a_t = \Delta t$ and $a_i = h, \forall i$, we attach a *link variable*, i.e. an approximation to the Wilson line between the points

$$U(n, n + a_\mu) = e^{i\int_n^{n+a_\mu} A\mathrm{d}z} \simeq e^{ia_\mu A_\mu(n+\frac{1}{2}a_\mu)} := U_\mu(n). \tag{17}$$

Observe that the link variable transforms under a gauge transformation as

$$U_\mu(n) \mapsto G^{-1}(n)U_\mu(n)G(n + a_\mu) = e^{ia_\mu A_\mu^G(n+\frac{1}{2}a_\mu)}, \tag{18}$$

where $A_\mu^G(n + \frac{1}{2}a_\mu)$ is the discretized version of $A_\mu(n + \frac{1}{2}a_\mu) + \partial_\mu \lambda(n)$.

We then approximate the Wilson loop by the product of link variables around an elementary plaquette (i.e. a face of a cube). Let this plaquette lie in the $\mu - \nu$ plane. We then get

$$U_{\mu\nu}(n) = U_\mu(n)U_\nu(n + a_\mu)U_\mu^\dagger(n + \nu)U_\nu^\dagger(n) = e^{ia_\mu a_\nu F_{\mu\nu}(n)}, \tag{19}$$

where $F_{\mu\nu}(n)$ is the components of the discretized version of the continuous field strenght tensor

$$F_{\mu\nu}(n) = \frac{1}{a_\mu}\delta_\mu A_\nu(n + \frac{1}{2}a_\nu) - \frac{1}{a_\nu}\delta_\nu A_\mu(n + \frac{1}{2}a_\mu), \tag{20}$$

where we have introduced a forward difference operator $\delta_\mu f(n) = f(n + a_\mu) - f(n)$. The equivalent backward difference operator is denoted by $\bar{\delta}_\mu f(n) = f(n) - f(n - a_\mu)$. From the transformation property of $U_\mu(n)$ we see that $U_{\mu\nu}(n)$ is gauge invariant. Observe that $F_{\mu\nu}(n)$ also satisfies the discrete Bianchi identity (the equivalent of (3) for the dual field strength)

$$\frac{1}{a_\lambda}\delta_\lambda F_{\mu\nu}(n) + \frac{1}{a_\mu}\delta_\mu F_{\nu\lambda}(n) + \frac{1}{a_\nu}\delta_\nu F_{\lambda\mu}(n) = 0. \tag{21}$$

for any μ, ν, λ.

With this at hand, one constructs the kinetic Maxwell action as the real part of the sum over all plaquettes of $U_{\mu\nu}$ with the appropriate weights, i.e.

$$S_{\text{kin}}[A] = \sum_n h(a)\Big(\alpha^2 \sum_i (1 - \cos(h\Delta t F_{0i}(n))) - \beta^2 \sum_{i<j}(1 - \cos(h^2 F_{ij}(n)))\Big), \tag{22}$$

where $h(a) = \Delta t h^3$, $\alpha = \frac{1}{\Delta t h}$, $\beta = \frac{1}{h^2}$ and latin indices take values between 1 and 3. By defining the electric and magnetic field components as $E_i = F_{0i}$ and $B_i = \frac{1}{2}\varepsilon_{ijk}F_{jk}$ where ε_{ijk} is the antisymmetric Levi-Civita tensor with $\varepsilon_{123} = 1$, this can be rewritten as

$$S_{\text{kin}}[A] = \sum_n h(a)\Big(\alpha^2 \sum(\mathbf{1} - \cos(h\Delta t \mathbf{E}(n))) - \beta^2 \sum(\mathbf{1} - \cos(h^2 \mathbf{B}(n)))\Big), \tag{23}$$

where we define the action of a real valued scalar function on a vector as

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(\mathbf{V}) = (f(V_x), f(V_y), f(V_z)) \in \mathbb{R}^3, \qquad \mathbf{V} \in \mathbf{R}^3, \tag{24}$$

$\sum \mathbf{V} = \sum_i V_i$ and $\mathbf{1} = (1, 1, 1)$. We note that the electric field has degrees of freedom on the edges of the mesh while the magnetic field has degrees of freedom on the faces/plaquettes of the mesh.

To complete the construction of the LGT action we need to add the source term JA, i.e we add

$$S_{source}[A] = \sum_n h(a) \left(-\rho(n)A^0(n) + \mathbf{j}(n) \cdot \mathbf{A}(n)\right), \tag{25}$$

to the kinetic action (22). The full LGT action for the Maxwell field with source is hence given as

$$S[A] = S_{\text{kin}}[A] + S_{source}[A]. \tag{26}$$

3.2 The finite difference equations from LGT

The discrete Euler-Lagrange equations corresponding to the action (26) are

$$\alpha\overline{\text{div}}(\sin(h\Delta t\mathbf{E})) = \rho, \tag{27}$$

and

$$\alpha\frac{1}{\Delta t}\bar{\delta}_t \sin(h\Delta t\mathbf{E}) - \beta\overline{\text{curl}}(\sin(h^2\mathbf{B}) = -\mathbf{j}, \tag{28}$$

where the discrete div- and curl-operators, $\overline{\text{div}}$ and $\overline{\text{curl}}$, are defined by backward Euler approximation of the derivatives, i.e. the adjoints with respect to the discrete L^2-scalar product of the forward Euler approximation of the gradient and the curl respectively. These two equations correspond to the inhomogeneous equation d$\tilde{F} = J$, and together with the discrete version of the Bianchi identity (21), the equivalent of d$F = 0$, they comprise the complete set of Maxwell's equations from LGT.

The equivalent of Theorem 1 holds in the discrete case as well, due to the gauge invariance, i.e.

Theorem 2 *Suppose* $(\mathbf{E}, \mathbf{B})$ *solves equation* (28) *at the lattice points on a time interval [0,T]. Suppose furthermore that the constraint* (27) *is satisfied at* $t = 0$ *and that the four-current is discrete divergence free, i.e.* $\frac{1}{\Delta t}\bar{\delta}_t\rho + \overline{\text{div}}\mathbf{j} = 0$. *Then the constraint* (27) *is satisfied at the lattice points for all* $n_t \in [0, T]$.

-Proof Calculate the backward difference of equation (27) with respect to time and use equation (28) to get the desired result.

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4 The classical Yee-scheme

The classical Yee-scheme [6] in a second order formulation consists of two inhomogeneous equations corresponding to the discrete Euler-Lagrange equations of the following action

$$\begin{aligned} S_{\text{Yee}}[A] &= \sum_n h(a) \left(\frac{1}{2}F_{0i}(n)^2 - \frac{1}{4}F_{ij}(n)^2 - \rho(n)A^0(n) + \mathbf{j}(n) \cdot \mathbf{A}(n)\right) \\ &= \sum_n h(a) \left(\frac{1}{2}\mathbf{E}(n) \cdot \mathbf{E}(n) - \frac{1}{2}\mathbf{B}(n) \cdot \mathbf{B}(n) - \rho(n)A^0(n) + \mathbf{j}(n) \cdot \mathbf{A}(n)\right), \end{aligned} \tag{29}$$

where $F_{\mu\nu}$ has the same form as in LGT, Eq. 20, and the discrete Bianchi identity, Eq. 21. The inhomogeneous equations consist of a constraint equation

$$\overline{\mathrm{div}}\mathbf{E} = \rho, \tag{30}$$

and an evolution equation

$$\frac{1}{\Delta t}\bar{\delta}_t \mathbf{E} - \overline{\mathrm{curl}}(\mathbf{B}) = -\mathbf{j}. \tag{31}$$

Observe the similarity with the LGT-scheme, i.e. with the substitutions

$$\alpha \sin(h\Delta t\mathbf{E}) \to \mathbf{E}, \qquad \beta \sin(h^2\mathbf{B}) \to \mathbf{B} \tag{32}$$

the LGT-scheme reduces to the Yee-scheme.

The Yee-scheme is, like the LGT-scheme, gauge invariant, i.e. invariant under the transformation $A_\mu \to A_\mu + 1/a_\mu \delta_\mu \lambda$, and due to this symmetry an equivalent of Theorem 2 holds.

4.1 The Yee energy

The Yee-energy is defined as [14]

$$H^m_{Yee} = \frac{1}{2}\sum_{\mathbf{n}} h^3 \left(\mathbf{E}^m(\mathbf{n})\cdot\mathbf{E}^m(\mathbf{n}) + \mathbf{B}^{m+1}(\mathbf{n})\cdot\mathbf{B}^m(\mathbf{n})\right) = \frac{1}{2}\left(\|\mathbf{E}^m\|^2_{\mathrm{L}^2} + \langle\mathbf{B}^{m+1},\mathbf{B}^m\rangle\right), \tag{33}$$

where the superscript m represents the time $t = m\Delta t$, and $\|\cdot\|_{\mathrm{L}^2}$ and $\langle\cdot,\cdot\rangle$ are the discrete L^2-norm and L^2-scalar product respectively, equivalent to the true L^2 product of the Finite Element vector fields that $\mathbf{E}$ and $\mathbf{B}$ represent. The similarity with the continuous expression, equation (13), is apparent.

This energy is constructed from the Yee-scheme, and the discrete time derivative of this expression will have the same structure as in the continuous case when applied to the Yee-fields, i.e.

$$\frac{1}{\Delta t}\bar{\delta}_t H^m_{Yee} = -\left\langle \mathbf{j}^m, \frac{1}{2}(\mathbf{E}^m + \mathbf{E}^{m-1})\right\rangle, \tag{34}$$

and we see that the energy is conserved in the absence of the source, $\mathbf{j} = 0$.

In order to use this energy in the convergence analysis, we need to introduce a CFL-condition on the lattice spacings to ensure the positivity of the Yee-energy. If we write

$$2\langle\mathbf{B}^{m+1},\mathbf{B}^m\rangle = \langle\mathbf{B}^{m+1},\mathbf{B}^{m+1}\rangle + \langle\mathbf{B}^m,\mathbf{B}^m\rangle - \langle\mathbf{B}^{m+1}-\mathbf{B}^{m+1},\mathbf{B}^{m+1}-\mathbf{B}^m\rangle, \tag{35}$$

and assume that the Bianchi identity is satisfied by $(\mathbf{E},\mathbf{B})$, we can rewrite the Yee-energy as

$$H^m_{Yee} = \frac{1}{2}\|\mathbf{E}^m\|^2_{\mathrm{L}^2} - \frac{\Delta t^2}{4}\|\mathrm{curl}_h\mathbf{E}^m\|^2_{\mathrm{L}^2} + \frac{1}{4}\left(\|\mathbf{B}^{m+1}\|^2_{\mathrm{L}^2} + \|\mathbf{B}^m\|^2_{\mathrm{L}^2}\right), \tag{36}$$

where the discrete curl-operator curl_h is defined by forward Euler approximation of the derivatives, the adjoint of $\overline{\mathrm{curl}}$. Obviously there exists a constant $C > 0$ such that $\|\mathrm{curl}_h\mathbf{E}^m\|^2_{\mathrm{L}^2} \le Ch^{-2}\|\mathbf{E}^m\|^2_{\mathrm{L}^2}$, and we see that by choosing

$$1 - C\frac{\Delta t^2}{2h^2} \ge \epsilon > 0 \tag{37}$$

we get the lower bound

$$H^m_{Yee}(\mathbf{E},\mathbf{B}) \ge \frac{\epsilon}{2}\|\mathbf{E}^m\|^2_{L^2} + \frac{1}{4}\left(\|\mathbf{B}^{m+1}\|^2_{L^2} + \|\mathbf{B}^m\|^2_{L^2}\right), \tag{38}$$

consisting of non-negative terms. The condition (37) is known as a CFL condition[15, 14].

5 Stability of LGT in the energy norm

In this section convergence of the LGT scheme will be shown. The convergence is proved in several steps, and the main ones are:

- We assume $\mathbf{j} \in \mathrm{L}^1([0,T];\mathrm{L}^2)$, and that the initial Yee-energy of the LGT-fields is strictly bounded by $K/2$, where K is a constant such that the energy of the continuous solution is strictly bounded by $K/2$ on the time interval $[0,T]$. We then show that there exists a time $T' > 0$ such that the Yee-energy of the LGT-fields is bounded by K on $[0,T']$ independently of h.
- Given the time T', we proceed to show that the Yee-energy of the difference between the LGT-fields and the Yee-fields approaches zero when the lattice spacing h goes to zero on $[0,T']$. This implies that the LGT-scheme is convergent on the time interval $[0,T']$ since the Yee-scheme is convergent [16, 14, 15].
- We ultimately want to show convergence up to a time T, and this is now done by iteration. Since the LGT-scheme is convergent on $[0,T']$, we can adjust the lattice spacing h such that the Yee-energy at time T' is again strictly bounded by $K/2$ and then prove convergence on $[0,2T']$. Proceed in this way to get convergence on $[0,T]$.

5.1 Boundedness of the LGT fields

In this subsection we show that the Yee-energy of the LGT-fields is bounded.

Lemma 1 *Suppose* $\mathbf{j} \in L^1([0,T];L^2)$ *and that the Yee-energy of the initial LGT-fields is strictly bounded by* $K/2$*, where* K *is a constant such that energy of the continuous solution is bounded by* $K/2$ *on the time interval* $[0,T]$*. Then there exists a time* $T_1 > 0$ *such that the Yee-energy of the LGT-fields is strictly bounded by* K *on the time interval* $[0,T_1]$ *provided that the lattice spacings are small enough and chosen as to fulfill a CFL condition, equation* (37)

-Proof Start out by writing

$$\sin(x) = x + r(x), \tag{39}$$

with the bounds

$$|r(x)| \leq \frac{1}{6}|x|^3, \qquad |r'(x)| \leq \frac{1}{2}|x|^2 \quad \Rightarrow \quad |r(x)-r(y)| \leq \frac{1}{2}(x^2+y^2)|x-y|, \qquad \forall x,y, \tag{40}$$

where r' means the derivative of r. With this at hand we can rewrite the evolution equation (28) as

$$\begin{aligned} &\frac{1}{\Delta t}\bar{\delta}_t \mathbf{E}^m - \overline{\mathrm{curl}}\mathbf{B}^m = \mathbf{c}^m \\ &\mathbf{c}^m = \frac{1}{h^2}\overline{\mathrm{curl}}(r(h^2\mathbf{B}^m)) - \frac{1}{h\Delta t}\frac{1}{\Delta t}\bar{\delta}_t r(h\Delta t\mathbf{E}^m) - \mathbf{j}^m := I_1 + I_2 + I_3, \end{aligned} \tag{41}$$

and from equation (34) we see that

$$\frac{1}{\Delta t}\bar{\delta}_t H^m_{Yee}(\mathbf{E},\mathbf{B}) = \left\langle \mathbf{c}^m, \frac{1}{2}(\mathbf{E}^m + \mathbf{E}^{m-1})\right\rangle. \tag{42}$$

In order to get stability of the scheme we hence need to control the right hand side which can be written as

$$\left\langle \mathbf{c}^m, \frac{1}{2}(\mathbf{E}^m + \mathbf{E}^{m-1})\right\rangle = \frac{1}{2}\Delta t\|\mathbf{c}^m\|^2_{L^2} + \left\langle \mathbf{c}^m, \mathbf{E}^{m-1}\right\rangle + \frac{1}{2}\Delta t\left\langle \mathbf{c}^m, \overline{\mathrm{curl}}\mathbf{B}^m\right\rangle \tag{43}$$

Since we have assumed that Δt and h satisfy a CFL-condition, $1 - C\frac{\Delta t^2}{2h^2} \geq \epsilon > 0$ where C is the constant such that $\|\text{curl}_h \mathbf{E}^m\|^2_{\text{L}^2} \leq Ch^{-2}\|\mathbf{E}^m\|^2_{\text{L}^2}$, we get the lower bound

$$H^m_{Yee}(\mathbf{E},\mathbf{B}) \geq \frac{\epsilon}{2}\|\mathbf{E}^m\|^2_{L^2} + \frac{1}{4}\left(\|\mathbf{B}^{m+1}\|^2_{L^2} + \|\mathbf{B}^m\|^2_{L^2}\right). \tag{44}$$

This implies the bounds

$$\begin{aligned} \|\mathbf{E}^{m-1}\|^2_{L^2} &\leq CH^{m-1}_{Yee}(\mathbf{E},\mathbf{B}) \\ \|\overline{\text{curl}}\mathbf{B}^m\|^2_{L^2} &\leq Ch^{-2}H^{m-1}_{Yee}(\mathbf{E},\mathbf{B}), \end{aligned} \tag{45}$$

where C is a constant. Hence, stability of 42 is controlled by the boundedness of $\|\mathbf{c}^m\|_{L^2}$. By Cauchy's inequality we get

$$\|\mathbf{c}^m\|^2_{L^2} \leq 3(\|I_1\|^2_{L^2} + \|I_2\|^2_{L^2} + \|I_3\|^2_{L^2}) := L_1 + L_2 + L_3. \tag{46}$$

With the estimate

$$h^{3/2}|\mathbf{B}^m(\mathbf{q})| \leq \|\mathbf{B}^m\|_{L^2} \leq C\sqrt{H^{m-1}_{Yee}(\mathbf{E},\mathbf{B})}, \tag{47}$$

we can immediately control L_1, i.e.

$$L_1 = 3\|\frac{1}{h^2}\overline{\text{curl}}(r(h^2\mathbf{B}^m))\|^2_{L^2} \leq Ch^6\|(\mathbf{B}^m)^3\|^2 \leq C(H^{m-1}_{Yee}(\mathbf{E},\mathbf{B}))^3, \tag{48}$$

where we use the notation $\mathbf{a}^n\mathbf{b}^m = (a^n_x b^m_x, a^n_y b^m_y, a^n_z b^m_z)$. The source term, L_3, contributes

$$L_3 = 3\|\mathbf{j}^m\|^2_{L^2}. \tag{49}$$

What remains is to control L_2. This term can further be divided into two parts

$$L_2 = 3\|\frac{1}{h\Delta t}\frac{1}{\Delta t}\bar{\delta}_t r(h\Delta t\mathbf{E}^m)\|^2 \leq Ch^4\Delta t^2(\|(\mathbf{E}^m)^3\|^2_{L^2} + \|(\mathbf{E}^{m-1})^3\|^2_{L^2}) := U_1 + U_2, \tag{50}$$

and with the estimate 47 in mind we easily see that

$$U_2 \leq C(H^{m-1}_{Yee}(\mathbf{E},\mathbf{B}))^3. \tag{51}$$

The only remaining part to estimate is U_1. The evolution equation (28) can be written as

$$\sin(h\Delta t\mathbf{E}^m) = \sin(h\Delta t\mathbf{E}^{m-1}) + \frac{\Delta t^2}{h}\overline{\text{curl}}(\sin(h^2\mathbf{B}^m)) - h\Delta t^2\mathbf{j}^m, \tag{52}$$

and since we may assume that H^{m-1}_{Yee} is bounded, inequality (47) and the equivalent estimate for the electric field guarantee that we can make the right hand side of the above equation smaller than one, so that

$$\mathbf{E}^m = \frac{1}{h\Delta t}\arcsin\left(\sin(h\Delta t\mathbf{E}^{m-1}) + \frac{\Delta t^2}{h}\overline{\text{curl}}\sin(h^2\mathbf{B}^m) - h\Delta t^2\mathbf{j}^m\right). \tag{53}$$

By use of the inequalities

$$|\arcsin(x)| \leq \frac{\pi}{2}|x|, \qquad |\sin(x)| \leq |x|, \tag{54}$$

we end up with the following estimate

$$|\mathbf{E}^m| \leq C(|\mathbf{E}^{m-1}| + |\mathbf{B}^m| + \Delta t|\mathbf{j}^m|). \tag{55}$$

By a similar argument as for U_2 we get

$$U_1 \leq C(H^{m-1}_{Yee}(\mathbf{E},\mathbf{B}))^3 + C(H^{m-1}_{Yee}(\mathbf{E},\mathbf{B}))^{1/2} + C\Delta t^2 \|\mathbf{j}^m\|^2_{L^2}. \tag{56}$$

Finally, combining (45) (46) (48) (49) (50) (51) (56) with Cauchy-Schwarz inequality give the estimate

$$\begin{aligned} \frac{1}{\Delta t}\bar{\delta}_t H^m_{Yee}(\mathbf{E},\mathbf{B}) &\leq f(H^{m-1}_{Yee}) + C\|\mathbf{j}^m\|_{L^2}(1 + \sqrt{H^{m-1}_{Yee}(\mathbf{E},\mathbf{B})}) \\ f(H^{m-1}_{Yee}) &:= C(H^{m-1}_{Yee}(\mathbf{E},\mathbf{B}))^2 + C(H^{m-1}_{Yee}(\mathbf{E},\mathbf{B}))^{1/2} + C\Delta t(H^{m-1}_{Yee}(\mathbf{E},\mathbf{B}))^3 + C\Delta t. \end{aligned} \tag{57}$$

Since we have assumed that $H^0_{Yee}(\mathbf{E},\mathbf{B}) < K/2$ we can deduce that

$$H^m_{Yee}(\mathbf{E},\mathbf{B}) < K, \qquad \text{provided} \qquad T_1 := m\Delta t < \frac{K/2 - C(1+\sqrt{K})\|\mathbf{j}\|_{L^1([0,T],L^2)}}{f(K)}, \tag{58}$$

and this concludes the proof.

■

When prooving convergence of the LGT-scheme in the next section we also need to estimate the Yee energy of the discrete time derivative of $\mathbf{E},\mathbf{B}$.

Lemma 2 *Let* $(\dot{\mathbf{E}},\dot{\mathbf{B}}) := \frac{1}{\Delta t}\delta_t(\mathbf{E},\mathbf{B})$, *where* $(\mathbf{E},\mathbf{B})$ *represents the LGT-fields. Suppose that the initial Yee-energy of* $(\dot{\mathbf{E}},\dot{\mathbf{B}})$ *is strictly bounded by* $K/2$ *where* K *is a constant such that the energy of the continuous solution is bounded by* $K/2$ *on the time interval* $[0,T]$. *Furthermore, assume that the Yee-energy of* $(\mathbf{E},\mathbf{B})$ *is bounded on* $[0,T_1]$ *and that* $\frac{1}{\Delta t}\delta_t\mathbf{j} \in L^1([0,T];L^2)$. *Then there exists a time* $T' > 0$ *(*$T' < T_1$*) such that the Yee-energy of* $(\dot{\mathbf{E}},\dot{\mathbf{B}})$ *is strictly bounded by* K *on* $[0,T']$ *provided that the lattice spacings are small enough and fulfill a CFL-condition.*

-*Proof* From the previous lemma, we immediately get

$$\frac{1}{\Delta t}\bar{\delta}_t H^m_{Yee}(\dot{\mathbf{E}},\dot{\mathbf{B}}) = \frac{1}{2}\Delta t\|\dot{\mathbf{c}}^m\|^2_{L^2} + \left\langle \dot{\mathbf{c}}^m, \dot{\mathbf{E}}^{m-1}\right\rangle + \frac{1}{2}\Delta t\left\langle \dot{\mathbf{c}}^m, \overline{\text{curl}}\dot{\mathbf{B}}^m\right\rangle \tag{59}$$

where $\mathbf{c}$ is defined in equation (41), and $\dot{\mathbf{c}} := \frac{1}{\Delta t}\delta_t\mathbf{c}$. Since Δt and h fulfill the CFL condition, the stability of (59) is controlled by the boundedness of $\|\dot{\mathbf{c}}^m\|^2_{L^2}$, and from equation (46) we see that

$$\|\dot{\mathbf{c}}^m\|^2_{L^2} \leq 3(\|\dot{I}_1\|^2_{L^2} + \|\dot{I}_2\|^2_{L^2} + \|\dot{I}_3\|^2_{L^2}) := L_1 + L_2 + L_3. \tag{60}$$

We controll L_1 by the boundedness of $\|\mathbf{B}\|_{L^2}$ and the mean value theorem, i.e.

$$\begin{aligned} L_1 = 3\|\frac{1}{h^2}\frac{1}{\Delta t}\overline{\text{curl}}(r(h^2\mathbf{B}^m))\|^2 &\leq C\frac{1}{h^6}\|h^6((\mathbf{B}^{m+1})^2 + (\mathbf{B}^m)^2)\dot{\mathbf{B}}^m\|^2 \\ &\leq CH^{m-1}_{Yee}(\dot{\mathbf{E}},\dot{\mathbf{B}}) \end{aligned} \tag{61}$$

The source term, L_3, contributes

$$L_3 = 3\|\frac{1}{\Delta t}\delta_t\mathbf{j}^m\|^2_{L^2}. \tag{62}$$

What remains is to controll L_2. This term is again divided in two, i.e.

$$\begin{aligned} L_2 &= 3\|\frac{1}{h\Delta t}\frac{1}{\Delta t}\delta_t\frac{1}{\Delta t}\bar{\delta}_t r(h\Delta t\mathbf{E}^m)\|^2 \\ &\leq C\frac{1}{h^2\Delta t^4}\left(\|\frac{1}{\Delta t}\delta_t r(h\Delta t\mathbf{E}^m)\|^2 + \|\frac{1}{\Delta t}\delta_t r(h\Delta t\mathbf{E}^{m-1})\|^2\right) := U_1 + U_2, \end{aligned} \tag{63}$$

and by a similar argument as we used for controlling L_1 we get $U_2 \leq CH^{m-1}_{Yee}(\dot{\mathbf{E}},\dot{\mathbf{B}})$. To estimate U_1 we use the evolution equation for $\mathbf{E}$ together with the mean value theorem to get the bound

$$|\dot{\mathbf{E}}^m| \leq C\left(|\dot{\mathbf{E}}^{m-1}| + |\dot{\mathbf{B}}^m| + \Delta t|\frac{1}{\Delta t}\delta_t\mathbf{j}^m|\right), \tag{64}$$

which implies

$$U_1 \leq C\|\dot{\mathbf{E}}^m\|^2 \leq CH^{m-1}_{Yee}(\dot{\mathbf{E}},\dot{\mathbf{B}}) + C. \tag{65}$$

Combining the above estimates we get

$$\begin{aligned} \frac{1}{\Delta t}\bar{\delta}_t H^m_{Yee}(\dot{\mathbf{E}},\dot{\mathbf{B}}) &\leq f(H^{m-1}_{Yee}) + C\|\frac{1}{\Delta t}\delta_t\mathbf{j}^m\|_{L^2}(1+\sqrt{H^{m-1}_{Yee}(\dot{\mathbf{E}},\dot{\mathbf{B}})}) \\ f(H^{m-1}_{Yee}) &:= CH^{m-1}_{Yee}(\dot{\mathbf{E}},\dot{\mathbf{B}}) + C\Delta t + C\sqrt{H^{m-1}_{Yee}(\dot{\mathbf{E}},\dot{\mathbf{B}})}. \end{aligned} \tag{66}$$

Since we have assumed that $H^0_{Yee}(\dot{\mathbf{E}},\dot{\mathbf{B}}) < K/2$ we can deduce that

$$H^m_{Yee}(\dot{\mathbf{E}},\dot{\mathbf{B}}) < K, \qquad \text{provided} \qquad T_2 := m\Delta t < \frac{K/2 - C(1+\sqrt{K})\|\frac{1}{\Delta t}\delta_t\mathbf{j}\|_{L^1([0,T],L^2)}}{f(K)}, \tag{67}$$

and this concludes the proof.

■

5.2 Estimates on the Yee energy of the difference between the Yee fields and the LGT fields

In this subsection we are going to bound the Yee energy of the difference between the Yee fields and the LGT fields by a constant times the lattice spacing on the time interval $[0,T']$. This will imply the convergence of the LGT-scheme on this time interval.

Lemma 3 *Suppose* $(\mathbf{E},\mathbf{B})$ *solves the evolution equation of the Yee-scheme, Eq. 31, and that* $(\tilde{\mathbf{E}},\tilde{\mathbf{B}})$ *solves the evolution equation of the LGT-scheme, Eq. 28. Furthermore suppose that the Yee-energy of both the Yee-fields, the LGT-fields and the discrete time derivatives are bounded on* $[0,T']$. *Then the Yee-energy of the difference* $(\Delta\mathbf{E},\Delta\mathbf{B}) = (\mathbf{E}-\tilde{\mathbf{E}},\mathbf{B}-\tilde{\mathbf{B}})$ *is bounded by the lattice spacing* h *on the time interval* $[0,T']$, *i.e.*

$$H^m_{Yee}(\Delta\mathbf{E},\Delta\mathbf{B}) \leq Cht, \qquad t = m\Delta t \in [0,T']. \tag{68}$$

-Proof From equations 31 and 41 we see that the evolution equation for $\Delta\mathbf{E} = \mathbf{E}-\tilde{\mathbf{E}}$ can be written as

$$\begin{aligned} \frac{1}{\Delta t}\bar{\delta}_t\Delta\mathbf{E}^m &= \overline{\text{curl}}\Delta\mathbf{B}^m + \mathbf{c}^m \\ \mathbf{c}^m &= \frac{1}{h\Delta t}\frac{1}{\Delta t}\bar{\delta}_t r(h\Delta t\tilde{\mathbf{E}}^m) - \frac{1}{h^2}\overline{\text{curl}}(r(h^2\tilde{\mathbf{B}}^m)), \end{aligned} \tag{69}$$

and we immediately get

$$\frac{1}{\Delta t}\bar{\delta}_t H^m_{Yee}(\Delta\mathbf{E},\Delta\mathbf{B}) = \left\langle \mathbf{c}^m, \frac{1}{2}(\Delta\mathbf{E}^m + \Delta\mathbf{E}^{m-1}) \right\rangle. \tag{70}$$

We write the right hand side as

$$\begin{aligned}\left\langle \mathbf{c}^m, \frac{1}{2}(\Delta\mathbf{E}^m + \Delta\mathbf{E}^{m-1}) \right\rangle = \frac{1}{h\Delta t}\left\langle \frac{1}{\Delta t}\bar{\delta}_t r(h\Delta t\tilde{\mathbf{E}}^m), \frac{1}{2}(\Delta\mathbf{E}^m + \Delta\mathbf{E}^{m-1}) \right\rangle - \\ \frac{1}{h^2}\left\langle \overline{\text{curl}}(r(h^2\tilde{\mathbf{B}}^m)), \frac{1}{2}(\Delta\mathbf{E}^m + \Delta\mathbf{E}^{m-1}) \right\rangle := I_1 + I_2,\end{aligned} \tag{71}$$

and analyse I_1 and I_2 separately.

In analysing I_1 we use that the Yee-energy of both the Yee-fields, the LGT-fields and the discrete time derivatives are bounded on $[0, T']$. This together with the mean value theorem imply that

$$\begin{aligned} I_1 &= \frac{1}{h\Delta t}\left\langle \frac{1}{\Delta t}\bar{\delta}_t r(h\Delta t\tilde{\mathbf{E}}^m), \frac{1}{2}(\Delta\mathbf{E}^m + \Delta\mathbf{E}^{m-1}) \right\rangle \\ &\le h^2\Delta t^2 \left\langle \left((\tilde{\mathbf{E}}^m)^2 + (\tilde{\mathbf{E}}^{m-1})^2\right)|\dot{\mathbf{E}}^{m-1}|, \frac{1}{2}\left|\Delta\mathbf{E}^m + \Delta\mathbf{E}^{m-1}\right| \right\rangle \le Ch \end{aligned} \tag{72}$$

In analysing I_2, we again use the boundedness of the Yee-energy of the various fields, the mean value theorem and a partial integration on the lattice, i.e.

$$\begin{aligned} I_2 &= -\frac{1}{h^2}\left\langle \overline{\text{curl}}(r(h^2\tilde{\mathbf{B}}^m)), \frac{1}{2}(\Delta\mathbf{E}^m + \Delta\mathbf{E}^{m-1}) \right\rangle \\ &\le Ch^4\left\langle |(\tilde{\mathbf{B}}^m)^3|, \left|\Delta\dot{\mathbf{B}}^m + \Delta\dot{\mathbf{B}}^{m-1}\right| \right\rangle \le Ch. \end{aligned} \tag{73}$$

This implies

$$\frac{1}{\Delta t}\bar{\delta}_t H^m_{Yee}(\Delta\mathbf{E},\Delta\mathbf{B}) \le Ch, \tag{74}$$

and we can conclude

$$H^m_{Yee}(\Delta\mathbf{E},\Delta\mathbf{B}) \le H^0_{Yee}(\Delta\mathbf{E},\Delta\mathbf{B}) + Cth = Cth, \qquad t := m\Delta t \in [0, T']. \tag{75}$$

■

We have now actually proved convergence of the LGT-scheme on the time interval $[0, T']$ since the Yee-scheme is convergent, but the restrictions we have on the initial conditions and the source $\mathbf{j}$ are not satisfactory. In our analysis so far, we have assumed that both the LGT-fields and their discrete time derivatives are in L^2 in space and with $\frac{1}{\Delta t}\delta_t\mathbf{j} \in L^1([0,T], L^2)$. What we would like is to have convergence in the energy norm, i.e. with initial data in L^2 ($(\mathbf{E},\mathbf{B})|_{t=0} \in \mathrm{L}^2$) and $\mathbf{j} \in L^1([0,T], L^2)$. We achieve this yet again with an energy estimate, and prove that the Yee-energy of the difference between two LGT-fields with different initial conditions can be estimated by the initial Yee-energy of the difference and the difference between the sources they are connected to.

Lemma 4 *Suppose $(\mathbf{E},\mathbf{B})$ and $(\tilde{\mathbf{E}},\tilde{\mathbf{B}})$ solve the LGT-scheme with different initial conditions and different sources, $\mathbf{j}$ and $\tilde{\mathbf{j}}$, and that the Yee-energy is bounded on the time interval $[0, T']$. Then the Yee-energy of the difference $(\Delta\mathbf{E},\Delta\mathbf{B}) = (\mathbf{E}-\tilde{\mathbf{E}},\mathbf{B}-\tilde{\mathbf{B}})$ is bounded by the initial value, $H^0_{Yee}(\Delta\mathbf{E},\Delta\mathbf{B})$ and the $L^1([0,T'], L^2)$-norm of the difference of the sources $||\mathbf{j}-\tilde{\mathbf{j}}||_{L^1([0,T'],L^2)}$, provided that the lattice spacings are small enough and fulfill a CFL-condition.*

-*Proof* From the previuos lemmas we immediately get

$$\frac{1}{\Delta t}\bar{\delta}_t H^m_{Yee}(\Delta\mathbf{E},\Delta\mathbf{B}) = \frac{1}{2}\Delta t\|\Delta\mathbf{c}^m\|^2_{L^2} + \left\langle \Delta\mathbf{c}^m, \Delta\mathbf{E}^{m-1}\right\rangle + \frac{1}{2}\Delta t\left\langle \Delta\mathbf{c}^m, \overline{\mathrm{curl}}\Delta\mathbf{B}^m\right\rangle, \tag{76}$$

with

$$\begin{aligned}\Delta\mathbf{c}^m &:= \frac{1}{h^2}\overline{\mathrm{curl}}\left(r(h^2\mathbf{B}^m) - r(h^2\tilde{\mathbf{B}}^m)\right) - \frac{1}{h\Delta t}\frac{1}{\Delta t}\bar{\delta}_t\left(r(h\Delta t\mathbf{E}^m) - r(h\Delta t\tilde{\mathbf{E}}^m)\right) - (\mathbf{j}^m - \tilde{\mathbf{j}}^m)\\ &:= I_1 + I_2 + I_3.\end{aligned} \tag{77}$$

As in the previous lemmas we have

$$\begin{aligned}\|\Delta\mathbf{E}^{m-1}\|^2_{L^2} &\leq CH^{m-1}_{Yee}(\Delta\mathbf{E},\Delta\mathbf{B})\\ \|\overline{\mathrm{curl}}(\Delta\mathbf{B}^m)\|^2_{L^2} &\leq Ch^{-2}H^{m-1}_{Yee}(\Delta\mathbf{E},\Delta\mathbf{B}),\end{aligned} \tag{78}$$

meaning that we need to control

$$\|\Delta\mathbf{c}^m)\|^2_{L^2} \leq 3(\|I_1\|^2_{L^2} + \|I_2\|^2_{L^2} + \|I_3\|^2_{L^2}) := L_1 + L_2 + L_3. \tag{79}$$

The term L_1 is controlled by the boundedness of the fields and the mean value theorem, i.e.

$$\begin{aligned}L_1 &= \frac{1}{h^4}\|\overline{\mathrm{curl}}\left(r(h^2\mathbf{B}^m) - r(h^2\tilde{\mathbf{B}}^m)\right)\|^2_{L^2} \leq Ch^6\|((\mathbf{B}^m)^2 + (\tilde{\mathbf{B}}^m)^2\Delta\mathbf{B}^m\|^2\\ &\leq CH^{m-1}_{Yee}(\Delta\mathbf{E},\Delta\mathbf{B})\end{aligned} \tag{80}$$

The approximation of L_2 is divided as in equation 50, i.e.

$$\begin{aligned}L_2 &= \frac{1}{h^2\Delta t^2}\|\frac{1}{\Delta t}\bar{\delta}_t\left(r(h\Delta t\mathbf{E}^m) - r(h\Delta t\tilde{\mathbf{E}}^m)\right)\|^2\\ &\leq Ch^2\Delta t^4\left(\|((\mathbf{E}^m)^2 + (\tilde{\mathbf{E}}^m)^2|\mathbf{E}^m - \tilde{\mathbf{E}}^m|\|^2 + \|((\mathbf{E}^{m-1})^2 + (\tilde{\mathbf{E}}^{m-1})^2|\mathbf{E}^{m-1} - \tilde{\mathbf{E}}^{m-1}|\|^2\right)\\ &:= U_1 + U_2.\end{aligned} \tag{81}$$

The term U_2 is approximated as as we did with L_1, and $U_2 \leq CH^{m-1}_{Yee}(\Delta\mathbf{E},\Delta\mathbf{B})$. In estimating U_1 we use the evolution equation to approximate

$$\begin{aligned}|\mathbf{E}^m - \tilde{\mathbf{E}}^m| \leq \frac{1}{h\Delta t}\Bigg|&\arcsin\left(\sin(h\Delta t\mathbf{E}^{m-1}) + \frac{\Delta t^2}{h}\overline{\mathrm{curl}}(\sin(h^2\mathbf{B}^m)) - h\Delta t^2\mathbf{j}^m\right) -\\ &\arcsin\left(\sin(h\Delta t\tilde{\mathbf{E}}^{m-1}) + \frac{\Delta t^2}{h}\overline{\mathrm{curl}}(\sin(h^2\tilde{\mathbf{B}}^m)) - h\Delta t^2\tilde{\mathbf{j}}^m\right)\Bigg|\\ \leq C\left(|\Delta\mathbf{E}^{m-1}| + h|\overline{\mathrm{curl}}(\Delta\mathbf{B}^m)| + \Delta t|\mathbf{j}^m - \tilde{\mathbf{j}}^m|\right).&\end{aligned} \tag{82}$$

This implies

$$\begin{aligned}U_1 &= Ch^4\Delta t^2\|((\mathbf{E}^m)^2 + (\tilde{\mathbf{E}}^m)^2|\mathbf{E}^m - \tilde{\mathbf{E}}^m|\|^2 \leq C\|\mathbf{E}^m - \tilde{\mathbf{E}}^m\|^2\\ &\leq CH^{m-1}_{Yee}(\Delta\mathbf{E},\Delta\mathbf{B}) + C\Delta t^2\|\mathbf{j}^m - \tilde{\mathbf{j}}^m\|^2_{L^2}\end{aligned} \tag{83}$$

The source term, L_3, contributes

$$L_3 \leq C\|\mathbf{j}^m - \tilde{\mathbf{j}}^m\|_{L^2}^2. \tag{84}$$

These estimates together with the Cauchy-Schwarz inequality give

$$\frac{1}{\Delta t}\bar{\delta}_t H_{Yee}^m(\Delta\mathbf{E}, \Delta\mathbf{B}) \leq C H_{Yee}^{m-1}(\Delta\mathbf{E}, \Delta\mathbf{B}) + C\|\mathbf{j}^m - \tilde{\mathbf{j}}^m\|_{L^2}, \tag{85}$$

and iterating on m gives

$$H_{Yee}^m(\Delta\mathbf{E}, \Delta\mathbf{B}) \leq (1 + \frac{Ct}{m})^m \left(H_{Yee}^0(\Delta\mathbf{E}, \Delta\mathbf{B}) + C\|\mathbf{j} - \tilde{\mathbf{j}}\|_{L^1([0,t],L^2)} \right), \quad t = m\Delta t \in [0, T']. \tag{86}$$

■

Since we can make both $H_{Yee}^0(\Delta\mathbf{E}, \Delta\mathbf{B})$ and $||\mathbf{j} - \tilde{\mathbf{j}}||_{L^1([0,t],L^2)}$ arbitrarily small, we can conclude from Lemma 1 - 4 that there exists a time $T' > 0$ such that the LGT-scheme converges in the energy norm on $[0, T']$. To get convergence up to a given time $T > 0$ we do as described in the introduction to this section. I.e., since the LGT-scheme is convergent on $[0, T']$ we can adjust the lattice spacing h such that the Yee-energy at time T' is again strictly bounded by $K/2$. Then we can repeat the process to show convergence on $[0, 2T']$. Proceed in this way to get convergence on $[0, T]$.

We summarize the result in a theorem:

Theorem 3 *Suppose the initial condition of the continuous problem is in L^2, i.e.*

$$(\mathbf{E}, \mathbf{B})|_{t=0} \in L^2, \tag{87}$$

and that the continuous source satisfies

$$\mathbf{j} \in L^1([0, T]; L^2). \tag{88}$$

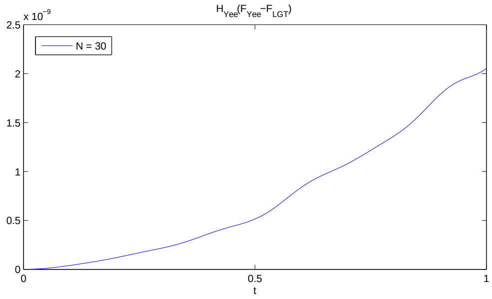
Suppose furthermore that the discrete initial condition and the discrete source converge exactly to the continuous ones. Then the discrete solution of the LGT scheme converges to the exact solution in the energy norm on [0,T].

6 Numerical results

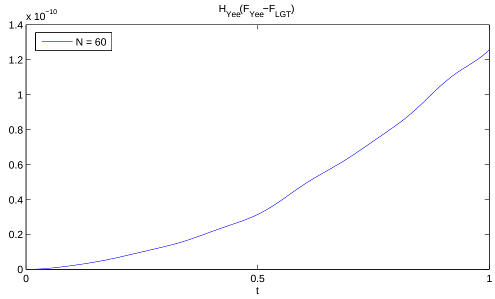
We have implemented both the LGT-scheme and the Yee-scheme, in a second order formulation, in the temporal gauge, $A_0 = 0$, on a space-time lattice with periodic boundary conditions in space and with no source term. The vector potential was initialized as a plane wave with the right periodicity, and with an initial energy of $H_{Yee}^0 = 0.5$. We used a lattice restricted to the spatial domain $[0, 1] \times [0, 1] \times [0, 1]$ and solved the equations in the time domain $t \in [0, 1]$. We used $N = 30$ ($N = 60$) lattice points in the spatial directions and $N_t = 100$ ($N_t = 200$) lattice points in the temporal direction.

In figure 1 and 2 the Yee-energy of the difference between the solutions is showed for $N = 30$ and $N = 60$ lattice points in the spatial directions. We see that the Yee-energy of the difference between the solutions in addition to be extremely small compared to the initial energy, behaves better than predicted, i.e. by halving the lattice spacing the energy difference is reduced by more than two. This has most likely to do with the choice of smooth initial conditions.

Figure 1: The Yee-energy of the difference between the LGT solution and the Yee solution as a function of time t



(a) The energy difference for $N = 30$ lattice points.



(b) The energy difference for $N = 60$ lattice points.

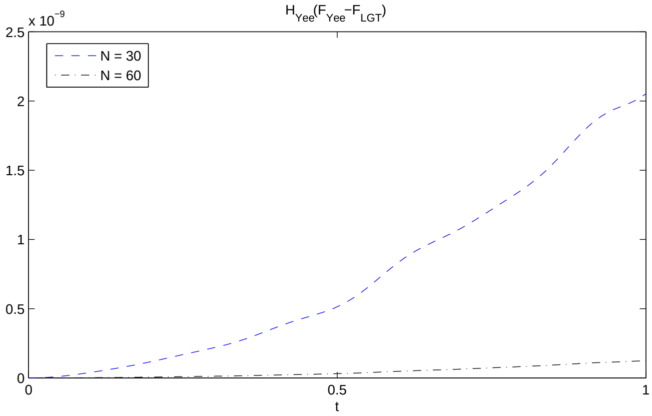


Figure 2: The Yee-energy of the difference between the LGT solution and the Yee solution as a function of time t for both $N = 30$ and $N = 60$ lattice points.

7 Conclusion

We have in this article studied discrete pure Maxwell theory from the perspective of Lattice Gauge Theory (LGT), and showed that the scheme is convergent in the energy norm by a comparison with the classical Yee-scheme. LGT is a theory originally arisen from high energy physics, and was constructed to discretize gauge theories in a gauge preserving way. LGT has therefore a much wider area of application than just Maxwell theory, and LGT has been used with some success for the Maxwell-Klein-Gordon equation [5].

The analysis of LGT for Maxwell theory can thus be viewed as a first step towards analysis of more complicated geometrical wave equations as for instance the Maxwell-Klein-Gordon equation or The Yang-Mills-Higgs equation from the LGT perspective.

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